

A REMARK ON THE PAPER “RANDOMIZING QUANTUM STATES: CONSTRUCTIONS AND APPLICATIONS”

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ABSTRACT. The concept of ε -randomizing quantum channels has been introduced by Hayden, Leung, Shor and Winter in connection with approximately encrypting quantum states. They proved using a discretization argument that sets of roughly $d \log d$ random unitary operators provide examples of such channels on \mathbf{C}^d . We show that a simple trick improves the efficiency of the argument and reduces the number of unitary operators to roughly d .

Since our argument is a minor modification of the original proof, we systematically refer the reader to [1] for introduction, background and applications of the notion of randomizing states.

Notation. On the space $\mathcal{B}(\mathbf{C}^d)$ of $d \times d$ complex matrices we consider the trace class norm $\|\cdot\|_1$ and the operator norm $\|\cdot\|_\infty$. Let also $\mathcal{D}(\mathbf{C}^d)$ be the convex set of mixed states (=positive elements of $\mathcal{B}(\mathbf{C}^d)$ with trace 1). The extreme points of $\mathcal{D}(\mathbf{C}^d)$ are pure states. We denote by C and c absolute numeric constants.

Definition. A quantum channel (= completely positive trace-preserving linear map) $R : \mathcal{B}(\mathbf{C}^d) \rightarrow \mathcal{B}(\mathbf{C}^d)$ is said to be ε -randomizing if for every state $\varphi \in \mathcal{D}(\mathbf{C}^d)$,

$$\left\| R(\varphi) - \frac{\text{Id}}{d} \right\|_\infty \leq \frac{\varepsilon}{d}.$$

Theorem. Let $(U_i)_{1 \leq i \leq N}$ be independent random matrices Haar-distributed on the unitary group $\mathcal{U}(d)$. Let $R : \mathcal{B}(\mathbf{C}^d) \rightarrow \mathcal{B}(\mathbf{C}^d)$ be the quantum channel defined by

$$R(\varphi) = \frac{1}{N} \sum_{i=1}^N U_i \varphi U_i^\dagger.$$

Assume that $0 < \varepsilon < 1$ and $N \geq Cd/\varepsilon^2 \cdot \log(1/\varepsilon)$. Then the channel R is ε -randomizing with nonzero probability.

As often with random constructions, we actually prove that the conclusion holds true with *large* probability. Let us quote two lemmas from [1].

Lemma (Lemma II.3 in [1]). Let φ, ψ be pure states on \mathbf{C}^d and $(U_i)_{1 \leq i \leq N}$ as before. Then for every $0 < \delta < 1$,

$$\mathbf{P} \left(\left| \frac{1}{N} \sum_{i=1}^N \text{Tr}(U_i \varphi U_i^\dagger \psi) - \frac{1}{d} \right| \geq \frac{\delta}{d} \right) \leq 2 \exp(-c\delta^2 N)$$

Lemma (Lemma II.4 in [1]). For $0 < \delta < 1$ there exists a set \mathcal{M} of pure states on \mathbf{C}^d with $|\mathcal{M}| \leq (5/\delta)^{2d}$, such that for every pure state φ on \mathbf{C}^d , there exists $\varphi_0 \in \mathcal{M}$ such that $\|\varphi - \varphi_0\|_1 \leq \delta$.

Proof of the theorem. Let A be the (random) quantity

$$A = \sup_{\varphi, \psi \in \mathcal{D}(\mathbf{C}^d)} \left| \frac{1}{n} \sum_{i=1}^n \text{Tr}(U_i \varphi U_i^\dagger \psi) - \frac{1}{d} \right|.$$

We must show that $\mathbf{P}(A \geq \frac{\varepsilon}{d}) < 1$. Let B be the restricted supremum over the set \mathcal{M}

$$B = \sup_{\varphi_0, \psi_0 \in \mathcal{M}} \left| \frac{1}{n} \sum_{i=1}^n \text{Tr}(U_i \varphi_0 U_i^\dagger \psi_0) - \frac{1}{d} \right|.$$

It follows from the lemmas that for δ to be determined later

$$\mathbf{P}\left(B \geq \frac{\delta}{d}\right) \leq (5/\delta)^{4d} \cdot 2 \exp(-c\delta^2 N).$$

Note that for any self-adjoint operators $a, b \in \mathcal{B}(\mathbf{C}^d)$

$$(1) \quad \left| \frac{1}{n} \sum_{i=1}^n \text{Tr}(U_i a U_i^\dagger b) \right| \leq \|a\|_1 \|b\|_1 \left(A + \frac{1}{d} \right).$$

By a convexity argument, the supremum in A can be restricted to pure states. Let φ, ψ be pure states and φ_0, ψ_0 in \mathcal{M} so that $\|\varphi - \varphi_0\|_1 \leq \delta, \|\psi - \psi_0\|_1 \leq \delta$. Then

$$\begin{aligned} \left| \frac{1}{n} \sum_{i=1}^n \text{Tr}(U_i \varphi U_i^\dagger \psi) - \frac{1}{d} \right| &\leq \left| \frac{1}{n} \sum_{i=1}^n \text{Tr}(U_i \varphi_0 U_i^\dagger \psi_0) - \frac{1}{d} \right| \\ &\quad + \left| \frac{1}{n} \sum_{i=1}^n \text{Tr}(U_i (\varphi - \varphi_0) U_i^\dagger \psi_0) \right| + \left| \frac{1}{n} \sum_{i=1}^n \text{Tr}(U_i \varphi U_i^\dagger (\psi - \psi_0)) \right|. \end{aligned}$$

Taking the supremum over φ, ψ and using twice (1), we get $A \leq B + 2\delta(A + 1/d)$, and so

$$A \leq \frac{1}{1 - 2\delta} \left(B + \frac{2\delta}{d} \right).$$

Choosing $\delta = \varepsilon/(3 + 2\varepsilon) \geq \varepsilon/5$ gives

$$\mathbf{P}\left(A \geq \frac{\varepsilon}{d}\right) \leq \mathbf{P}\left(B \geq \frac{\delta}{d}\right) \leq 2 \left(\frac{25}{\varepsilon}\right)^{4d} \exp(-c\varepsilon^2 N/25).$$

The last quantity is less than 1 provided $N \geq Cd/\varepsilon^2 \cdot \log(1/\varepsilon)$ for some constant C .

Remark. One checks (using the value $c = (6 \ln 2)^{-1}$ from [1]) that for d large enough, the constant in our theorem can be chosen to $C = 150$. This is presumably far from optimal.

REFERENCES

- [1] P. Hayden, D. Leung, P. W. Shor and A. Winter, Randomizing quantum states: constructions and applications, *Comm. Math. Phys.* **250** (2004), 371–391.

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